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APRIL 1954

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On Systems of Ordinary, Non-linear Differential Equations Involving Periodic Given Functions with a Small Period

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PREPARED UNDER
CONTRACT No. N6ori-201, TASK ORDER 1
WITH THE
OFFICE OF NAVAL RESEARCH

IMM-NYU 20
April 1954

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Institute of Mathematical Sciences

ON SYSTEMS OF ORDINARY, NON-LINEAR DIFFERENTIAL EQUATIONS INVOLVING PERIODIC GIVEN FUNCTIONS WITH A SMALL PERIOD

by
Joel Franklin

This report represents results obtained at the Institute of Mathematical Sciences, New York University, under the auspices of Contract N6ori-201, Task Order 1, with the Office of Naval Research.

New York, 1954

On Systems of Ordinary, Non-linear Differential Equations Involving Periodic Given Functions with a Small Period

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1. Introduction. In this paper systems of ordinary differential equations

$$(1.1) \qquad \frac{dx_\nu}{dt} = f_\nu(t, x_1, \ldots, x_n) \qquad (\nu=1,\ldots,n)$$

are considered in which the given functions f_ν are periodic functions of time, t, with a small period $\varepsilon > 0$. It is shown that, if ε is sufficiently small, then the solutions of the system (1.1) are, in general, well approximated over a finite time-interval by appropriate solutions of the system

$$(1.2) \qquad \frac{dy_\nu}{dt} = g_\nu(y_1, \ldots, y_n) \qquad (\nu=1,\ldots,n),$$

where the functions g_ν are the time-averages

$$g_\nu(y_1, \ldots, y_n) = \frac{1}{\varepsilon} \int_0^\varepsilon f_\nu(t, y_1, \ldots, y_n)dt \qquad (\nu=1,\ldots,n)$$

Numerical estimates are given which, it is hoped, may be useful to the applied mathematician. The manner in which these estimates may be used is illustrated in the last section by a concrete application to a problem of a type occurring in mathematical physics.

2. Assumptions. For any vector $z = (z_1, z_2, \ldots, z_n)$ with real components we may define the norm $|z|$ as either

$$(2.1) \qquad |z| = \sum |z_\nu| \quad \text{or} \quad |z| = \left\{ \sum z_\nu^2 \right\}^{1/2} .$$

Let either one of these definitions be chosen and used throughout. Let T and A be fixed positive numbers. Now let a vector-function $f(t,x) = (f_1(t,x_1,\ldots,x_n),\ldots,f_n(t,x_1,\ldots,x_n))$ with real-valued components be defined in the range $0 \leqq t \leqq T$, $|x| \leqq A$. We assume that $f(t,x)$ is periodic:

$$(2.2) \qquad f(t+\varepsilon,x) = f(t,x),$$

where ε is a positive number less than T. And we assume that $f(t,x)$ is bounded:

$$(2.3) \qquad |f(t,x)| \leqq M .$$

Furthermore, $f(t,x)$ should be a continuous function of t and x except for at most a finite number of values of t independent of x. Finally, we assume that $f(t,x)$ satisfies a Lipschitz condition:

$$(2.4) \qquad |f(t,x) - f(t,y)| \leqq L|x-y|,$$

where L is a constant > 0. (Bear in mind that the estimates (2.3) and (2.4) are required to hold only in the given range of definition of $f(t,x)$.)

Now for $|x| \leqq A$ we may define the function

$$(2.5) \qquad g(x) = \frac{1}{\varepsilon} \int_0^{\varepsilon} f(t,x)\, dt.$$

It follows from (2.3) and (2.4) that g(x) is bounded and that g(x) satisfies a Lipschitz condition:

$$(2.6) \qquad |g(x)| \leqq M', \quad |g(x)-g(y)| \leqq L'|x-y|,$$

where $M' \leqq M$ and $L' \leqq L$.

Finally, we define a scalar function $\rho(t)$ as follows:

$$(2.7) \qquad \rho(t) = \alpha(t) + L \int_0^t e^{L(t-\tau)}\alpha(\tau)d\tau ,$$

where

$$(2.8) \qquad \alpha(t) = \frac{1}{4} (L+L')M'\varepsilon[\frac{t}{\varepsilon}] + (M+M') (\frac{t}{\varepsilon} - [\frac{t}{\varepsilon}]);$$

here the symbol $[\frac{t}{\varepsilon}]$ denotes the greatest integer not greater than $\frac{t}{\varepsilon}$. We note that $\rho(0) = 0$; that for $t \geqq 0$ the function $\rho(t)$ is continuous except for $t = \varepsilon, 2\varepsilon, 3\varepsilon, \ldots$; that $\rho(t) \geqq 0$ for $t \geqq 0$; and that for $0 \leqq t \leqq T$ the function $\rho(t)$ is less than some constant independent of ε. We shall discuss this function in greater detail in Section 4.

Before proceeding to the theorem, we make the familiar remark that the differential system (1.1) is equivalent to the integral

equation

$$x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau ,$$

where x_0 is an arbitrary constant initial vector.
However, it is to be understood that the continuous solution $x = u(t)$ need not be differentiable at the values of t for which f is discontinuous.

3. The Theorem. First we prove two simple lemmas.

Lemma 1. The function $\rho(t)$ defined in (2.7) satisfies the integral equation

$$(3.1) \qquad \rho(t) = \alpha(t) + L \int_0^t \rho(\tau)\, d\tau ,$$

where $\alpha(t)$ is defined in (2.8).

Proof: The right-hand side of (3.1) is equal to

$$(3.2) \quad \alpha(t) + L \int_0^t \alpha(\tau)\, d\tau + L^2 \int_0^t \int_0^\tau e^{L(\tau-\zeta)} \alpha(\zeta)\, d\zeta\, d\tau .$$

Because α is a continuous function except for simple jump-discontinuities, the order of integration in (3.2) may be reversed, so that the last term in (3.2) equals

$$L^2 \int_0^t \int_\zeta^t e^{L(\tau-\zeta)} \alpha(\zeta)\, d\tau\, d\zeta = L \int_0^t e^{L(t-\zeta)} \alpha(\zeta)\, d\zeta - L \int_0^t \alpha(\zeta)\, d\zeta .$$

Hence, the expression (3.2) is equal to

$$\alpha(t) + L \int_0^t e^{L(t-\zeta)}\alpha(\zeta)d\zeta = \rho(t) ,$$

as required.

Lemma 2. Let $\sigma(t)$ be any integrable function satisfying the inequality

$$(3.3) \quad \sigma(t) \leqq \alpha(t) + L \int_0^t \sigma(\tau)d \qquad (0 \leqq t \leqq T_0) .$$

Then

$$(3.4) \qquad \sigma(t) \leqq \rho(t) \qquad (0 \leqq t \leqq T_0) .$$

Proof: Subtracting (3.3) from (3.1), we find

$$(3.5) \quad \rho(t) - \sigma(t) \geqq L \int_0^t \{\rho(\tau) - \sigma(\tau)\} d\tau \quad (0 \leqq t \leqq T_0) .$$

Since the right-hand side is continuous, the function $\rho(t) - \sigma(t)$ has a finite lower bound, say $-\mu$. Hence, using (3.5) repeatedly, we find

$$\rho(t) - \sigma(t) \geqq - \frac{1}{k!} L^k\mu t^k \qquad (k=1,2,3,\ldots).$$

Letting $k \longrightarrow \infty$, we find $\rho(t) - \sigma(t) \geqq 0$, as required.

Theorem. Assume the conditions of Section 2. Let x_0 be a

constant vector with $|x_0| < A$. Assume that for $0 \leqq t \leqq T$ the equation

$$(3.6) \qquad y(t) = x_0 + \int_0^t g(y(\tau))d\tau$$

has the solution $y = v(t)$, and that for some $a < A$ we have

$$(3.7) \qquad |v(t)| \leqq a \qquad (0 \leqq t \leqq T) .$$

Let T^* be the largest value of $t' \leqq T$ for which the inequality

$$(3.8) \qquad \max_{0 \leqq t \leqq t'} \rho(t) \leqq \frac{A-a}{\varepsilon}$$

holds. (Thus, $0 < T^* \leqq T$.) Then the equation

$$(3.9) \qquad x(t) = x_0 + \int_0^t f(\tau, x(\tau))d\tau$$

has a solution $x = u(t)$ for $0 \leqq t \leqq T^*$, and $u(t)$ satisfies the inequality

$$(3.10) \qquad |u(t) - v(t)| \leqq \varepsilon\rho(t) \qquad (0 \leqq t \leqq T^*) .$$

<u>Remark</u>. The sense of the theorem is as follows: If $\varepsilon > 0$ is very small, then $\frac{A-a}{\varepsilon}$ will be very large. Then we should find T^* large, probably even as great as T. Then the inequality (3.10) shows that v(t) closely approximates u(t) over a rather large range. But the solution $y = v(t)$ of the initial-value problem

$$\frac{d}{dt}\, y(t) = g(y(t)) \;, \qquad y(0) = x_0$$

is generally comparatively easy to discuss, since the function $g(y)$ does not depend explicitly on t, and since $g(y)$, being the time average of $f(t,y)$, should be in some sense a smoother function than $f(t,y)$. These observations will be illustrated by the example in Section 5.

Proof of the theorem. By the classical existence-and-uniqueness theorem for systems of ordinary differential equations, we know that the equation (3.9) has a unique solution $x = u(t)$ in a positive neighborhood of $t = 0$, and that this solution may be continued as long as t and the vector $x = u(t)$ do not leave the range $0 \leqq t \leqq T$, $|x| \leqq A$. Let us say that the solution $u(t)$ may be thus continued for $0 \leqq t \leqq T_0$, where $0 < T_0 \leqq T$. Then in this interval we have

$$(3.11)\qquad \begin{aligned} u(t) - v(t) &= \int_0^t \{f(\tau,u(\tau)) - g(v(\tau))\}\, d\tau \\ &= \int_0^t \{f(\tau,\, u(\tau)) - f(\tau,v(\tau))\} d\tau + \int_0^t \{f(\tau,v(\tau)) - g(v(\tau))\} d\tau \,. \end{aligned}$$

According to the Lipschitz condition (2.4), the first of the two integrals in (3.11) is not greater in norm than

$$(3.12)\qquad L \int_0^t |u(\tau) - v(\tau)|\, d\tau \,.$$

Letting $N = [\frac{t}{\varepsilon}]$ and $\delta = t - \varepsilon N$, we may write the second integral

in (3.11) in the form

$$(3.13)\quad \sum_{k=0}^{N-1} \int_{k\varepsilon}^{(k+1)\varepsilon} \{f(\tau, v(\tau)) - g(v(\tau))\} d\tau + \int_{t-\delta}^{t} \{f(\tau, v(\tau)) - g(v(\tau))\} d\tau .$$

Since $|f| \leqq M$ and $|g| \leqq M'$, the last integral in (3.13) has norm not greater than $(M+M')\delta$. Now the $k^{\underline{th}}$ integral in (3.13) may be written as

$$(3.14)\quad \int_{k\varepsilon}^{(k+1)\varepsilon} \{f(\tau, v((k+\tfrac{1}{2})\varepsilon)) - g(v((k+\tfrac{1}{2})\varepsilon))\}\, d\tau$$

$$+ \int_{k\varepsilon}^{(k+1)\varepsilon} \{f(\tau, v(\tau)) - f(\tau, v((k+\tfrac{1}{2})\varepsilon)) + g(v((k+\tfrac{1}{2})\varepsilon)) - g(v(\tau))\}\, d\tau .$$

By the periodicity of f, the first integral in (3.14) equals

$$\int_0^{\varepsilon} f(\tau, v((k+\tfrac{1}{2})\varepsilon))d\tau - \varepsilon g(v((k+\tfrac{1}{2})\varepsilon)) \quad ,$$

which equals zero, by the definition (2.5) of g. According to the Lipschitz conditions (2.4), (2.6), the second integral in (3.14) has norm not greater than

$$(3.15)\quad (L+L') \int_{k\varepsilon}^{(k+1)\varepsilon} |v(\tau) - v((k+\tfrac{1}{2})\varepsilon)|\, d\tau .$$

But since v satisfies the equation (3.6), and since $|g| \leqq M'$, the integrand in (3.15) is $\leqq M' |\tau - (k+\tfrac{1}{2})\varepsilon|$, so that the expression (3.15) is $\leqq \frac{1}{4}(L+L')M'\varepsilon^2$. In summary, we have found

$$|u(t)-v(t)| \leqq L \int_0^t |u(\tau)-v(\tau)|d\tau + \tfrac{1}{4}(L+L')M'N\varepsilon^2 + (M+M')\delta ,$$

or

$$(3.16) \qquad |u(t)-v(t)| \leqq \varepsilon\alpha(t) + L \int_0^t |u(\tau)-v(\tau)|d\tau ,$$

where $\alpha(t)$ is the function defined in (2.8). We may, therefore, replace $\sigma(t)$ in Lemma 2 by $\varepsilon^{-1}|u(t)-v(t)|$, so that (3.4) becomes

$$(3.17) \qquad |u(t)-v(t)| \leqq \varepsilon\, \rho(t) \qquad (0 \leqq t \leqq T_0) .$$

Now it remains only to show that we may take T_0 equal to the number T^* defined in the statement of the theorem. According to the remarks made at the beginning of this proof, it is sufficient to show that every continuous vector $x(t)$ satisfying the inequality

$$(3.18) \qquad |x(t) - v(t)| \leqq \varepsilon\, \rho(t) \qquad (0 \leqq t \leqq T^*)$$

must lie in the range

$$(3.19) \qquad |x(t)| \leqq A \qquad (0 \leqq t \leqq T^*) .$$

But, by the definition of T^*, we have

$$\rho(t) \leqq \frac{A-a}{\varepsilon} \qquad (0 \leqq t \leqq T^*) .$$

Since $|v| \leqq a$, it follows that the inequality (3.18) implies the required inequality (3.19). We know, therefore, that the solution $x = u(t)$ of equation (3.9) exists on the whole interval $0 \leqq t \leqq T^*$. Replacing T_0 by T^* in the inequality (3.17), we see that the proof of the theorem is complete.

4. Estimation of $\rho(t)$ and T^*. In practice it is often convenient to have simple estimates for the function $\rho(t)$ defined in (2.7) and for the number T^* defined in the statement of the theorem. We shall find an estimate for $\rho(t)$ which preserves and makes obvious the most important properties of that function. However, our estimate for T^*, though simple, does not preserve the most important property of T^*, namely that $T^* > 0$. (Of course, $T^* > 0$ because $A-a > 0$, $\rho(0) = 0$, and $\rho(t)$ is continuous for $0 \leqq t < \varepsilon$.) Finally in this section, we shall find an exact representation for $\rho(t)$ which may sometimes be useful.

To estimate $\rho(t)$, we first note from (2.8) that

$$(4.1) \qquad \alpha(\tau) \leqq \frac{1}{4}(L+L')M'\tau + M+M' .$$

Using this inequality in the integral in (2.7), we find

$$(4.2) \qquad \rho(t) \leqq \alpha(t) + \frac{L+L'}{4L} M' \left\{ e^{Lt}-(1+Lt) \right\} + (M+M')(e^{Lt}-1) ,$$

or

$$(4.3) \qquad \rho(t) \leqq \frac{1}{4}(L+L')M'\varepsilon[\frac{t}{\varepsilon}] + (M+M')(\frac{t}{\varepsilon} - [\frac{t}{\varepsilon}])$$

$$+ \frac{L+L'}{4L} M' \left\{e^{Lt}-(1+Lt)\right\} + (M+M')(e^{Lt}-1) \ .$$

To estimate T^*, we first find a cruder estimate for $\rho(t)$ by using the inequality (4.1) in the inequality (4.2). We thus find

$$(4.4) \qquad \rho(t) \leqq (\frac{L+L'}{4L}M'+M+M')e^{Lt} - \frac{L+L'}{4L} M' \ .$$

Therefore, certainly $\rho(t) \leqq \frac{A-a}{\varepsilon}$ provided that the right-hand side of (4.4) is $\leqq \frac{A-a}{\varepsilon}$, i.e. provided that $t \leqq T'$, where

$$(4.5) \qquad T' = \frac{1}{L}\left\{\log \frac{1}{\varepsilon} + \log \frac{4L(A-a) + \varepsilon M'(L+L')}{4LM + 5LM' + L'M'}\right\} \ .$$

It follows that-

$$(4.6) \qquad T^* \geqq T' \text{ if } T' \leqq T; \quad T^* + T \text{ if } T' > T \ .$$

Finally we compute $\rho(t)$ exactly. For convenience, set

$$(4.7) \qquad \alpha_1 = \frac{M+M'}{\varepsilon} \ , \ \alpha_2 = \frac{1}{4}(L+L')M'\varepsilon - (M+M'),$$

so that

$$(4.8) \qquad \alpha(t) = \alpha_1 t + \alpha_2[\frac{t}{\varepsilon}] \ ,$$

$$(4.9)\quad \rho(t) = \alpha(t) + Le^{Lt}\left(\alpha_1 \int_0^t e^{-Lt}\tau d\tau \;+ \alpha_2 \int_0^t e^{-L\tau}[\tfrac{\tau}{\varepsilon}]d\tau\right) .$$

The first integral in (4.9) is elementary. By the theorem of integration by parts for Stieltjes integrals, the second integral equals

$$-\frac{1}{L}[\tfrac{t}{\varepsilon}]e^{-Lt} + \frac{1}{L}\sum_{0<k\varepsilon\leqq t} e^{-Lk\varepsilon} = -\frac{1}{L}[\tfrac{t}{\varepsilon}]e^{-Lt} + \frac{1}{L}\, e^{-L\varepsilon} \cdot \frac{1-e^{-L\varepsilon[\frac{t}{\varepsilon}]}}{1-e^{-L\varepsilon}} .$$

We thus obtain the representation

$$\rho(t) = \alpha(t) + \frac{\alpha_1}{L}\left\{e^{Lt}-(1+Lt)\right\} - \alpha_2\left\{[\tfrac{t}{\varepsilon}] + e^{Lt}\cdot \frac{1-e^{-L\varepsilon[\frac{t}{\varepsilon}]}}{1-e^{L\varepsilon}}\right\},$$

or

$$(4.10)\quad \rho(t) = \frac{\alpha_1}{L}\,(e^{Lt}-1) - \alpha_2 e^{Lt} \cdot \frac{1-e^{-L\varepsilon[\frac{t}{\varepsilon}]}}{1-e^{L\varepsilon}} .$$

5. <u>Example</u>. For $0 \leqq t \leqq 2$, we consider a moving particle whose displacement, x_1, is governed by the non-linear differential equation

$$(5.1)\quad \frac{d^2x_1}{dt^2} + p(t)x_1 + \left\{\sin^2 200\pi t + p(t)\right\} x_1^3 = 0 ,$$

where $p(t)$ is the function defined by the scheme

$$(5.2)\quad p(t) = \begin{cases} 0 & t = 0 \\ 1 & 0 < t < \frac{1}{400} \\ 0 & t = \frac{1}{400} \\ -1 & \frac{1}{400} < t < \frac{1}{200} \end{cases}$$

and by the requirement of periodicity:

$$(5.3) \qquad p(t) = p(t + \tfrac{1}{200}) \ .$$

The initial velocity of the particle is known to be zero. We denote the initial displacement of the particle by the letter ξ. Without loss of generality, we may assume that $\xi > 0$, since, if x_1 is a solution of the equation (5.1), so is $-x_1$.

The problem is to find an upper bound, ξ_0, for the initial displacement ξ such that, if $\xi \leqq \xi_0$, then

$$(5.4) \qquad |x_1(t)| \leqq \frac{1}{10} \quad \text{for } 0 \leqq t \leqq 2 \ .$$

Of course, we know from general considerations that such a number ξ_0 exists; the practical problem is to find a value for ξ_0 which is not too small in comparison with the number $\frac{1}{10}$, which appears in the inequality (5.4). We shall find, by means of the preceding theory, that it is sufficient to take

$$(5.5) \qquad \xi_0 = 0{\cdot}08976 \ .$$

We shall also find a closer description of the displacement x_1 and the velocity $\dot{x}_1$. In fact, if $\xi \leqq \xi_0$, then for $0 \leqq t \leqq 2$

$$(5.6) \quad \xi - \tfrac{1}{4}\xi^3 t^2 - \frac{1}{200}\,\rho(t) \leqq x_1(t) \leqq \xi + \frac{1}{200}\,\rho(t) \ ,$$

$$(5.7)\quad -\frac{1}{2}\,\xi^3 t - \frac{1}{200}\,\rho(t) \leqq \dot{x}_1(t) \leqq \frac{1}{200}\,\rho(t) \quad ,$$

where

$$(5.8)\ \rho(t) \leqq 0.000258\,[200t] + 0.202\,(200t - [200t])$$
$$+ 0.486\left\{e^{1.06t} - (1+1.06t)\right\} + 0.202(e^{1.06t}-1).$$

To apply the theory to the present case, we first set

$$\varepsilon = \frac{1}{200}\ ,\quad A = \frac{1}{10}\ ,\quad T = 2 \quad .$$

Setting $x_2 = \dot{x}_1$, we have

$$(5.9)\quad f(t,x) = (x_2,\ -p(t)x_1 - \left\{\sin^2 200\pi t + p(t)\right\}\ x_1^3) \quad .$$

Taking the time-average, we find

$$g(y) = (y_2\ ,\ -\frac{1}{2}\,y_1^3)\ .$$

Next, we have

$$x_o = (\xi,\ 0)\ .$$

For this problem we shall use the second definition (2.1) of the norm, namely $|z| = |(z_1,\ z_2)| = \sqrt{z_1^2 + z_2^2}$. Accordingly , we have

$$|g(y)|^2 = y_2^2 + \frac{1}{4}y_1^6 \leqq \max_{\theta} \ (10^{-2}\sin^2\theta + \frac{1}{4}\ 10^{-6}\cos^6\theta) \ .$$

But

$$10^{-2}\sin^2\theta + \frac{1}{4}\ 10^{-6}\cos^6\theta = 10^{-2}-10^{-2}\cos^2\theta + \frac{1}{4}\ 10^{-6}\cos^6\theta \leqq 10^{-2} \ .$$

Hence,

$$M' = \frac{1}{10} \ .$$

To find L', we write

$$|g(x)-g(y)|^2 \leqq (x_2 - y_2)^2 + \frac{1}{4}(x_1 - y_1)^2(x_1^2+|x_1y_1|+ y_1^2)^2$$

$$\leqq (x_2 - y_2)^2 + \frac{1}{4}(x_1 - y_1)^2\left(\frac{3}{100}\right)^2 \leqq |x-y|^2 \ .$$

Therefore,

$$L' = 1 \ .$$

From (5.9) we now find

$$|f(t,x)|^2 \leqq x_2^2 + x_1^2(1+2x_1^2)^2$$

$$\leqq \max_{\theta} \left\{10^{-2}\sin^2\theta + 10^{-2}\cos^2\theta + 10^{-2}(0.04+0.0004)\right\}$$

$$= 0.010404 = (0.102)^2.$$

Hence, we may take

$$M = 0.102 \quad .$$

Next,

$$\begin{aligned} |f(t,x)-f(t,y)|^2 &\leqq (x_2-y_2)^2 + (x_1-y_1)^2 \left\{1+2(x_1^2+|x_1y_1|+y_1^2)\right\}^2 \\ &\leqq (1.06)^2|x-y|^2 \quad , \end{aligned}$$

so that we may set

$$L = 1.06$$

In summary, we have the following data:

$$(5.10) \qquad \varepsilon = 0.005, \quad A = 0.1, \quad T = 2 \ ,$$

$$L = 1.06, \ M = 0.102, \ L' = 1, \ M' = 0.1 \ .$$

According to the estimate (4.6), to ensure that $T^* = T = 2$, it is sufficient that the number T' defined in (4.5) be $\geqq 2$. Replacing T' by 2 and = by $\leqq$ in (4.5), and solving the resulting inequality for a, we see that it is sufficient to have

$$(5.11) \quad a \leqq A - \frac{\varepsilon}{4L}\left\{e^{2L}(4LM+5LM'+L'M') - (L+L')M'\right\}.$$

Computation with the data (5.10) gives (5.11) the form

$$(5.12) \qquad a \leq 0.08980 \quad .$$

It now remains only to find the number ξ_o in the range

$$(5.13) \qquad 0 < \xi_o < 0.08980$$

such that, if $0 < \xi \leq \xi_o$, then the solution of the initial-value problem

$$(5.14) \qquad \frac{d^2y_1}{dt^2} + \frac{1}{2} y_1^3 = 0 \ , \ y_1(0) = \xi, \ \dot{y}_1(0) = 0$$

will satisfy the condition

$$(5.15) \qquad |y(t)| \leq 0.08980 \qquad (0 \leq t \leq 2) \ .$$

But the problem (5.14) is very easy to discuss. Since $y_1(0) > 0$, there must be a number $t_o > 0$ for which

$$y_1(t) > 0 \qquad (0 \leq t < t_o \ ) \quad .$$

It now follows from (5.14) that

$$(5.16) \qquad \ddot{y}_1(t) < 0, \ \dot{y}_1(t) \leq 0, \ y_1(t) \leq \xi \quad (0 \leq t < t_o) \quad .$$

From (5.14) and from the third inequality in (5.16) it follows that

$$(5.17)\quad \ddot{y}_1(t) \geqq -\tfrac{1}{2}\xi^3,\ \dot{y}_1(t) \geqq -\tfrac{1}{2}\xi^3 t,\ y_1(t) \geqq \xi - \tfrac{1}{4}\xi^3 t^2 (0 \leqq t < t_0).$$

But, since $0 < \xi \leqq \xi_0 < 0.1$, we have

$$(5.18)\quad \xi - \frac{1}{4}\xi^3 t^2 > 0 \text{ for } 0 \leqq t \leqq \frac{2}{\xi} > 20 .$$

We may, therefore, take for t_0 any number $\leqq 20$.

It now follows from the inequalities (5.16) and (5.17) that

$$|y(t)|^2 \leqq \xi^2 + (-\tfrac{1}{2}\xi^3 t)^2 \leqq \xi^2 + \xi^6 \quad (0 \leqq t \leqq 2) .$$

Since $0 < \xi < 0.1$, this implies

$$|y(t)| < \xi\sqrt{1.0001} \quad (0 \leqq t \leqq 2) .$$

Therefore, to ensure that $a \leqq 0.08980$, it is sufficient to take $0 < \xi \leqq \xi_0$, where ξ_0 is determined by the equation

$$\xi_0 \sqrt{1.0001} = 0.08980 .$$

Computing, we find the result (5.5).

To obtain the inequalities (5.6) and (5.7) it is only necessary to consider the function $\rho(t)$ discussed in Section 3. Using the data (5.10) in the estimate (4.3), we find the inequality (5.8) for $\rho(t)$. But, according to (3.10), we have in our case

$$|x(t) - y(t)| \leqq \frac{1}{200}\rho(t) \quad (0 \leqq t \leqq 2).$$

It follows that

$$(5.19)\quad \begin{aligned} y_1(t) - \frac{1}{200}\,\rho(t) &\leqq x_1(t) \leqq y_1(t) + \frac{1}{200}\,\rho(t) \\ \dot{y}_1(t) - \frac{1}{200}\,\rho(t) &\leqq \dot{x}_1(t) \leqq \dot{y}_1(t) + \frac{1}{200}\,\rho(t) \end{aligned} \quad .$$

Applying the inequalities (5.17) to the left-hand sides of (5.19), and applying the inequalities (5.16) to the right-hand sides, we obtain the required inequalities (5.6) and (5.7) .

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by the University's Office of Publications and Printing